# Newton's Third Law in the Framework of Special Relativity

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**Abstract**

Newton's third law states that any action is countered by a reaction of equal magnitude but opposite direction. The total force in a system not affected by external forces is thus zero. However, according to the principles of relativity a signal can not propagate at speeds exceeding the speed of light. Hence the action cannot be generated at the same time with the reaction because the information about the action has to reach the affected object and the affected object still needs additional time to react on the source, hence the total force cannot be null at a given time. The following analysis provides for a better understanding of the ways natural laws would behave within the framework of Special Relativity, and on how this understanding may be used for practical purposes.

# 1 Introduction

Among the major achievements of Sir Isaac Newton is the formulation of Newton's third law stating that any action is countered by a reaction of equal magnitude but opposite direction [1, 2]. The total force in a system not affected by external forces is thus zero. This law has numerous experimental verifications and seems to be one of the corner stones of physics. However, by the middle of the nineteenth century Maxwell has formulated the laws of electromagnetism in his famous four partial differential equations [3, 4, 5] which were formulated in their current form by Oliver Heaviside [6]. One of the consequences of these equations is that an electromagnetic signal cannot travel at speeds exceeding that of light. This was later used by Albert Einstein [7, 4, 5] (among other things) to formulate his special theory of relativity which postulates that the speed of light is the maximal allowed velocity in nature. According to the principles of relativity no signal (even if not electromagnetic) can propagate at superluminal velocities. Hence an action and its reaction cannot be generated at the same time because the knowledge about the action has to reach the affected object and the affected object still needs additional time to react on the source. Thus the total force cannot be null at a given time. In consequence, by not holding rigorously the simultaneity of action and reaction Newton's third law cannot hold in exact form but only as an approximation. Moreover, the total force within a system that is not acted upon by an external force would not be rigorously null since the action and reactions are not able to balance each other and the total force on a system which is not affected by an external force in not null in an exact sense.

Most locomotive systems of today are based on open systems. A rocket sheds exhaust gas to propel itself, a speeding bullet generates recoil. A car pushes the road with the same force that is used to accelerate it, the same is true regarding the interaction of a plane with air and of a ship with water. However, the above relativistic considerations suggest's a new type of motor which is not based on a open system but rather on a closed one.

In this paper we discuss the force between two current carrying coils. Two mathematical treatments will be given, in one we consider an instantaneous action at a distance, here Newton's third law hold and the total electromagnetic force will be shown to be equal to zero. Then we consider the dynamic, electro-magnetic condition where the reaction to an action cannot occur before having the action-generated information reach the affected object, thus bringing about a non-zero resultant.

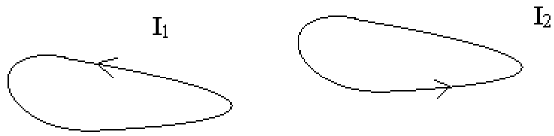


Figure 1: Two current loops.

## 2 The Magneto - Static Condition

Consider a wire segment of length $d\vec{l_2}$ located at $\vec{x}_2$ carrying a current $I_2$ (see figure 1). The magneto-static field at point $\vec{x}_1$ due to the line segment at $\vec{x}_2$ is [4]:

$$d\vec{B}(\vec{x}_1) = \frac{\mu_0}{4\pi} I_2 \frac{d\vec{l_2} \times \vec{x}_{12}}{|\vec{x}_{12}|^3}. \tag{1}$$

In the above $\vec{x}_{12} = \vec{x}_1 - \vec{x}_2$ and $\mu_0 = 4\pi 10^{-7}$ is the vacuum magnetic permeability. The magnetic field at point $\vec{x}_1$ originating from the entire current loop 2 is given by the integral:

$$\vec{B}(\vec{x}_1) = \frac{\mu_0}{4\pi} I_2 \oint \frac{d\vec{l_2} \times \vec{x}_{12}}{|\vec{x}_{12}|^3}. \tag{2}$$

Now if at point $\vec{x}_1$ there is a wire segment of length $d\vec{l_1}$ carrying a current $I_1$, then according to the law of Lorentz [4] the wire segment will experience a force:

$$d\vec{F}_{12} = I_1 d\vec{l_1} \times \vec{B}(\vec{x}_1). \tag{3}$$

The entire current loop 1 will experience a total force:

$$\vec{F}_{12} = I_1 \oint d\vec{l_1} \times \vec{B}(\vec{x}_1). \tag{4}$$

Inserting equation (2) into equation (4) we obtain the following expression for the force on the current loop 1:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{d\vec{l_1} \times (d\vec{l_2} \times \vec{x}_{12})}{|\vec{x}_{12}|^3}. \tag{5}$$

Using standard vector identities one can show that:

$$d\vec{l_1} \times (d\vec{l_2} \times \vec{x}_{12}) = (d\vec{l_1} \cdot \vec{x}_{12}) d\vec{l_2} - (d\vec{l_1} \cdot d\vec{l_2}) \vec{x}_{12}, \tag{6}$$

hence equation (5) can be written in the form:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \left[ \oint \oint \frac{(\vec{dl_1} \cdot \vec{x}_{12})\vec{dl_2}}{|\vec{x}_{12}|^3} - \oint \oint \frac{(\vec{dl_1} \cdot \vec{dl_2})\vec{x}_{12}}{|\vec{x}_{12}|^3} \right]. \tag{7}$$

However since:

$$\frac{\vec{x}_{12}}{|\vec{x}_{12}|^3} = -\vec{\nabla}_{\vec{x}_1} \frac{1}{|\vec{x}_{12}|} \tag{8}$$

in which $\vec{\nabla}_{\vec{x}_1}$ is the gradient operator with respect to the $\vec{x}_1$ coordinate. Hence we obtain:

$$\oint \frac{\vec{dl_1} \cdot \vec{x}_{12}}{|\vec{x}_{12}|^3} = -\oint \vec{dl_1} \cdot \vec{\nabla}_{\vec{x}_1} \frac{1}{|\vec{x}_{12}|} = 0. \tag{9}$$

Thus the first integral in equation (7) does not contribute and we are left with the expression:

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{(\vec{dl_1} \cdot \vec{dl_2})\vec{x}_{12}}{|\vec{x}_{12}|^3}. \tag{10}$$

From the above it is easy to calculate the magneto-static force applied by current loop 1 on current loop 2 by changing indices $1 \leftrightarrow 2$:

$$\vec{F}_{21} = -\frac{\mu_0}{4\pi} I_2 I_1 \oint \oint \frac{(\vec{dl_2} \cdot \vec{dl_1})\vec{x}_{21}}{|\vec{x}_{21}|^3}. \tag{11}$$

However, since:

$$\vec{x}_{21} = \vec{x}_2 - \vec{x}_1 = -(\vec{x}_1 - \vec{x}_2) = -\vec{x}_{12}, \qquad |\vec{x}_{21}| = |\vec{x}_{12}| \tag{12}$$

We obtain:

$$\vec{F}_{21} = -\vec{F}_{12}. \tag{13}$$

And the total force on two loop system is null:

$$\vec{F}_T = \vec{F}_{12} + \vec{F}_{21} = 0. \tag{14}$$

The above result is independent of the geometry of the loops and is in complete agreement with Newton's third law.

## 3 The Dynamic Electromagnetic Condition

Let us now consider the more general time dependent case. Since for time dependence one can not separate the magnetic and electric forces which are connected through Maxwell's equations. We will consider both the electric and magnetic parts of the Lorenz force $\vec{F}_{21}$. Let us suppose that the electric field $\vec{E}$ and magnetic field $\vec{B}$ are created by current loop 1 and acts upon current loop 2. Since a conductive loop that carries a neutral charge will contain both ions and free electrons (in equal amounts) we will have:

$$\vec{F}_{21} = \int d^3x_2 \rho_{i2}(\vec{E} + \vec{v}_{i2} \times \vec{B}) + \int d^3x_2 \rho_{e2}(\vec{E} + \vec{v}_{e2} \times \vec{B}). \tag{15}$$

In the above we integrate over the entire volume of current loop 2. $\rho_{i2}$ and $\rho_{e2}$ are the ion charge density and electron charge density respectively, $\vec{v}_{i2}$ and $\vec{v}_{e2}$ are the ion velocity field and electron velocity field respectively. We will assume that the magnetization and polarization of the medium are small and therefore we neglect corrections to the Lorentz force suggested in [8]. Since the amounts of ions and free electrons are equal in a neutral conducting loop, we write:

$$\rho_{i2} = -\rho_{e2}. \tag{16}$$

Thus the electric terms in the above force equation cancel and we are left with:

$$\vec{F}_{21} = \int d^3x_2 \rho_{i2} \vec{v}_{i2} \times \vec{B} + \int d^3x_2 \rho_{e2} \vec{v}_{e2} \times \vec{B}. \tag{17}$$

In the laboratory frame the ions being at rest we have: $\vec{v}_{i2} = 0$. Thus we arrive at the expression:

$$\vec{F}_{21} = \int d^3x_2 \rho_{e2} \vec{v}_{e2} \times \vec{B}. \tag{18}$$

Introducing the current density: $\vec{J}_2 = \rho_{e2}\vec{v}_{e2}$ we obtain the expression:

$$\vec{F}_{21} = \int d^3x_2 \vec{J}_2 \times \vec{B}. \tag{19}$$

Now, let us consider the coil that generates the magnetic field. The magnetic field can be written as follows in terms of its vector potential [4]:

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{20}$$

If the field is generated by a current density $\vec{J}_1$ in coil 1[1] we can solve for the vector potential and obtain the result [4]:

$$\vec{A}(\vec{x}_2) = \frac{\mu_0}{4\pi}\int d^3x_1 \frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}, \qquad \vec{R} \equiv \vec{x}_{12}, \qquad t_{ret} \equiv t - \frac{R}{c}. \tag{21}$$

In the above $t$ is time and $c$ is the speed of light in vacuum. Combining equation (21) with equation (20) we arrive at the result:

$$\vec{B}(\vec{x}_2) = \vec{\nabla}_{\vec{x}_2} \times \vec{A}(\vec{x}_2) = \frac{\mu_0}{4\pi}\int d^3x_1 \vec{\nabla}_{\vec{x}_2} \times \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right). \tag{22}$$

However, notice that[2]:

$$\vec{\nabla}_{\vec{x}_2} \times \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right) = \vec{\nabla}_{\vec{x}_2} R \times \partial_R \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right). \tag{23}$$

Since:

$$\vec{\nabla}_{\vec{x}_2} R = -\frac{\vec{R}}{R} \tag{24}$$

And:

$$\partial_R \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right) = -\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R^2} - \frac{\partial_t \vec{J}_1(\vec{x}_1, t_{ret})}{Rc}. \tag{25}$$

Hence:

$$\vec{\nabla}_{\vec{x}_2} \times \left(\frac{\vec{J}_1(\vec{x}_1, t_{ret})}{R}\right) = \frac{\vec{R}}{R^3} \times \left(\vec{J}_1(\vec{x}_1, t_{ret}) + \left(\frac{R}{c}\right) \partial_t \vec{J}_1(\vec{x}_1, t_{ret})\right). \tag{26}$$

Inserting equation (26) into equation (22) we arrive at the result:

$$\vec{B}(\vec{x}_2) = \frac{\mu_0}{4\pi}\int d^3x_1 \frac{\vec{R}}{R^3} \times \left(\vec{J}_1(\vec{x}_1, t_{ret}) + \left(\frac{R}{c}\right) \partial_t \vec{J}_1(\vec{x}_1, t_{ret})\right). \tag{27}$$

The current density in a thin-conductor loop can be expressed in terms of the loop's current as follows:

$$\int d^3x_1 g(\vec{x}_1) \vec{J}_1(\vec{x}_1, t) = \int dl_1 g(\vec{x}_1) \int dA_1 \vec{J}_1(\vec{x}_1, t) = \int d\vec{l}_1 g(\vec{x}_1) I_1(t). \tag{28}$$

[1]It is assumed that coil 1 has zero total charge density as is the case for coil 2 and any other neutral coil.

[2]We use the notation $\partial_y \equiv \frac{\partial}{\partial y}$.

In the above $dA_1$ is a cross section area element of the loop and $g(\vec{x}_1)$ is an arbitrary function. In terms of the result given in equation (28) one may write equation (27) as:

$$\vec{B}(\vec{x}_2, t) = \frac{\mu_0}{4\pi} \oint \frac{\vec{R}}{R^3} \times d\vec{l}_1 \left( I_1(t_{ret}) + \left(\frac{R}{c}\right) \partial_t I_1(t_{ret}) \right). \qquad (29)$$

Inserting equation (29) into equation (19) we arrive at the result:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} \int d^3x_2 \vec{J}_2 \times \oint \frac{\vec{R}}{R^3} \times d\vec{l}_1 \left( I_1(t_{ret}) + \left(\frac{R}{c}\right) \partial_t I_1(t_{ret}) \right). \qquad (30)$$

Assuming that current loop 2 has also a small cross section area and using the same argument as in equation (28) we arrive at the result:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_2(t) \oint d\vec{l}_2 \times \oint \frac{\vec{R}}{R^3} \times d\vec{l}_1 \left( I_1(t_{ret}) + \left(\frac{R}{c}\right) \partial_t I_1(t_{ret}) \right). \qquad (31)$$

## 3.1 The Quasi-Static Approximation

In the quasi-static approximation we assume that $\tau = \frac{R}{c}$ is small and can be neglected. Under the same approximation $t_{ret} \simeq t$. Neglecting all terms of order $\tau$ we arrive at the result:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_1(t) I_2(t) \oint d\vec{l}_2 \times \oint \frac{\vec{R}}{R^3} \times d\vec{l}_1. \qquad (32)$$

However, since $\vec{R} = \vec{x}_{12}$ we write:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_1(t) I_2(t) \oint \oint d\vec{l}_2 \times \left(\frac{\vec{x}_{12}}{|\vec{x}_{12}|^3} \times d\vec{l}_1\right). \qquad (33)$$

Repeating the same type of arguments as in equation (6) we obtain:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_1(t) I_2(t) \left[ \oint \oint -\frac{(d\vec{l}_2 \cdot \vec{x}_{12}) d\vec{l}_1}{|\vec{x}_{12}|^3} + \oint \oint \frac{(d\vec{l}_1 \cdot d\vec{l}_2) \vec{x}_{12}}{|\vec{x}_{12}|^3} \right]. \qquad (34)$$

And using equation (9) with $1 \leftrightarrow 2$ the first integral vanishes:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_1(t) I_2(t) \oint \oint \frac{(d\vec{l}_1 \cdot d\vec{l}_2) \vec{x}_{12}}{|\vec{x}_{12}|^3}. \qquad (35)$$

Or:

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi} I_1(t) I_2(t) \oint \oint \frac{(d\vec{l}_1 \cdot d\vec{l}_2)\vec{x}_{12}}{|\vec{x}_{12}|^3}. \tag{36}$$

This is the quasi-static analogue of equation (10) and is identical to the static case except for the fact that now the currents are dependent on time. The same conclusions may be drawn as in the static case. Newton's third law holds regardless of the shape of the coils involved and the total force is null.

This should come as no surprise to the reader since in the quasi-static approximation we totally neglect the time it takes a signal to propagate from coil 1 to coil 2 assuming that this time is zero. Different results are obtained when $\tau$ is not neglected as we will see in the next subsection.

### 3.2 The Case of a Finite $\tau$

Consider the current $I(t_{ret}) = I(t - \frac{R}{c})$, if $\frac{R}{c}$ is small but not zero one can write a Taylor series expansion around $t$ in the form:

$$I(t_{ret}) = I(t - \frac{R}{c}) = \sum_{n=0}^{\infty} \frac{I^{(n)}(t)}{n!}(-\frac{R}{c})^n. \tag{37}$$

In the above $I^{(n)}(t)$ is the derivative of order $n$ of $I(t)$. Inserting equation (37) into equation (21) and taking into account equation (28) we obtain:

$$\vec{A}(\vec{x}_2) = \frac{\mu_0}{4\pi} \oint d\vec{l}_1 \frac{I_1(t_{ret})}{R} = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{I_1^{(n)}(t)}{n!} \oint d\vec{l}_1 \frac{1}{R}(-\frac{R}{c})^n \tag{38}$$

Denoting $g_n(R) = \frac{1}{R}(-\frac{R}{c})^n$ and inserting equation (38) into equation (20) we obtain:

$$\vec{B}(\vec{x}_2) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{I_1^{(n)}(t)}{n!} \oint \vec{\nabla}_{\vec{x}_2} \times (d\vec{l}_1 g_n(R)) \tag{39}$$

Now writing equation (19) in terms of equation (39) and taking into account equation (28) we have:

$$\vec{F}_{21} = \oint d\vec{l}_2 I_2(t) \times \vec{B} = \frac{\mu_0}{4\pi} I_2(t) \sum_{n=0}^{\infty} \frac{I_1^{(n)}(t)}{n!} \oint \oint d\vec{l}_2 \times (\vec{\nabla}_{\vec{x}_2} g_n(R) \times d\vec{l}_1). \tag{40}$$

Using a well known vector identity:

$$d\vec{l}_2 \times (\nabla_{\vec{x}_2} g_n(R) \times d\vec{l}_1) = \vec{\nabla}_{\vec{x}_2} g_n(R)(d\vec{l}_2 \cdot d\vec{l}_1) - d\vec{l}_1(d\vec{l}_2 \cdot \vec{\nabla}_{\vec{x}_2} g_n(R)) \tag{41}$$

We can write:

$$\oint \oint d\vec{l_2} \times (\vec{\nabla}_{\vec{x}_2} g_n(R) \times d\vec{l_1}) = \oint \oint \vec{\nabla}_{\vec{x}_2} g_n(R)(d\vec{l_2} \cdot d\vec{l_1}) - \oint d\vec{l_1} \oint d\vec{l_2} \cdot \vec{\nabla}_{\vec{x}_2} g_n(R) \tag{42}$$

but since $\oint d\vec{l_2} \cdot \vec{\nabla}_{\vec{x}_2} g_n(R) = 0$ equation (40) can be written as:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_2(t) \sum_{n=0}^{\infty} \frac{I_1^{(n)}(t)}{n!} \oint \oint \vec{\nabla}_{\vec{x}_2} g_n(R)(d\vec{l_2} \cdot d\vec{l_1}). \tag{43}$$

However, since:

$$\begin{aligned} \vec{\nabla}_{\vec{x}_2} g_n(R) &= (-\frac{1}{c})^n \vec{\nabla}_{\vec{x}_2} R^{n-1} = (-\frac{1}{c})^n (n-1) R^{n-2} \vec{\nabla}_{\vec{x}_2} R \\ &= (-\frac{1}{c})^n (1-n) R^{n-3} \vec{R} \end{aligned} \tag{44}$$

The force takes the form:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_2(t) \sum_{n=0}^{\infty} \frac{I_1^{(n)}(t)}{n!} (-\frac{1}{c})^n (1-n) \oint \oint R^{n-3} \vec{R} (d\vec{l_2} \cdot d\vec{l_1}). \tag{45}$$

We note that there is no first order contribution to the force. Hence the next contribution to the force after the quasi-static term is second order. Let us define the dimensionless geometrical factor $\vec{K}_{21n}$ as:

$$\vec{K}_{21n} = \frac{1}{h^n} \oint \oint R^{n-3} \vec{R} (d\vec{l_2} \cdot d\vec{l_1}) = -\vec{K}_{12n}. \tag{46}$$

in the above $h$ is some characteristic distance between the coils. In terms of $\vec{K}_{21n}$ we can write equation (45) as:

$$\vec{F}_{21} = \frac{\mu_0}{4\pi} I_2(t) \sum_{n=0}^{\infty} \frac{I_1^{(n)}(t)}{n!} (-\frac{h}{c})^n (1-n) \vec{K}_{21n}. \tag{47}$$

The force due to coil 2 that acts on coil 1 is:

$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1(t) \sum_{n=0}^{\infty} \frac{I_2^{(n)}(t)}{n!} (-\frac{h}{c})^n (1-n) \vec{K}_{12n}. \tag{48}$$

The total force on the system is thus:

$$\vec{F}_T = \vec{F}_{12} + \vec{F}_{21} = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{(1-n)}{n!} (-\frac{h}{c})^n \vec{K}_{12n} \left( I_1(t) I_2^{(n)}(t) - I_2(t) I_1^{(n)}(t) \right). \tag{49}$$

We note that the quasi-static term $n = 0$ does not contribute to the sum nor does the $n = 1$ term. The fact that the retarded field "corrects" itself to first order in order to "mimic" a non retarded field was already noticed by Feynman [5]. Hence we can write:

$$\vec{F}_T = \frac{\mu_0}{4\pi} \sum_{n=2}^{\infty} \frac{(1-n)}{n!} (-\frac{h}{c})^n \vec{K}_{12n} \left( I_1(t) I_2^{(n)}(t) - I_2(t) I_1^{(n)}(t) \right). \tag{50}$$

We conclude that in general Newton's third law is not satisfied, taking the leading non-vanishing terms in the above sum we obtain:

$$\begin{aligned} \vec{F}_T &\cong -\frac{\mu_0}{8\pi} (\frac{h}{c})^2 \vec{K}_{122} \left( I_1(t) I_2^{(2)}(t) - I_2(t) I_1^{(2)}(t) \right) \\ &= -\frac{\mu_0}{8\pi} (\frac{h}{c})^2 \vec{K}_{122} I_1(t) I_2(t) \left( \frac{I_2^{(2)}(t)}{I_2(t)} - \frac{I_1^{(2)}(t)}{I_1(t)} \right). \end{aligned} \tag{51}$$

### 3.3 Two Circular Loops

In the following we will calculate the geometrical factor $\vec{K}_{12n}$ for a simple geometry of two circular loops of unit radius and a unit distance located one atop the other (see figure 2). According to equation (46) we have in the case $h = 1$[3]

$$\vec{K}_{12n} = -\oint \oint R^{n-3} \vec{R} (d\vec{l_2} \cdot d\vec{l_1}). \tag{52}$$

Every point on loop 1 and 2 has the coordinates $\vec{x}_1 = (\cos\theta_1, \sin\theta_1, 1)$ and $\vec{x}_2 = (\cos\theta_2, \sin\theta_2, 0)$, hence:

$$\vec{R} = \vec{x}_{12} = \vec{x}_1 - \vec{x}_2 = (\cos\theta_1 - \cos\theta_2, \sin\theta_1 - \sin\theta_2, 1) \tag{53}$$

And thus:

$$R^2 = 3 - 2\cos(\theta_1 - \theta_2) \tag{54}$$

The line elements can be calculated as follows:

$$\begin{aligned} d\vec{l_1} &= d\theta_1 \hat{\theta}_1 = d\theta_1(-\sin\theta_1, \cos\theta_1, 0) \\ d\vec{l_2} &= d\theta_2 \hat{\theta}_2 = d\theta_2(-\sin\theta_2, \cos\theta_2, 0) \end{aligned} \tag{55}$$

Hence:

$$d\vec{l_1} \cdot d\vec{l_2} = d\theta_1 d\theta_2 \cos(\theta_1 - \theta_2) \tag{56}$$

[3] In arbitrary units since $\vec{K}_{12n}$ is dimensionless.

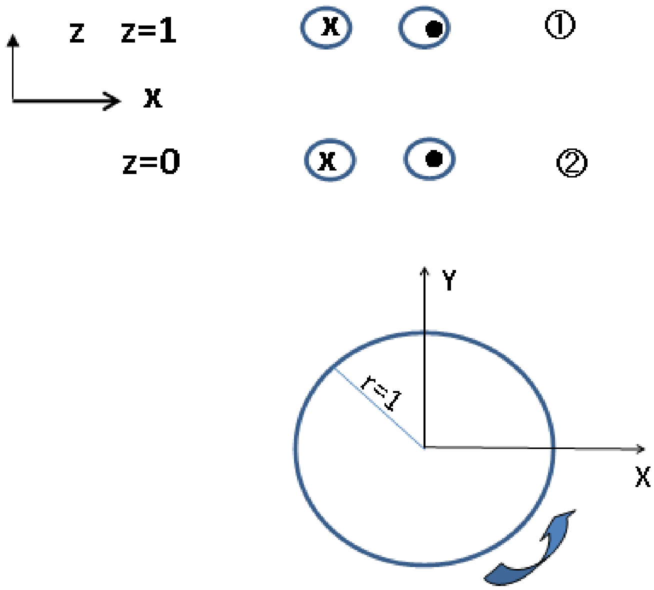


Figure 2: Two circular loops of radius 1 and distance 1. Both x-z and x-y cross sections are shown.

Inserting equation (56), equation (54) and equation (53) into equation (52) we arrive at:

$$\vec{K}_{12n} = -\int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\theta_1 \cos(\theta_1 - \theta_2) \left(\sqrt{3 - 2\cos(\theta_1 - \theta_2)}\right)^{n-3} \vec{R}(\theta_1, \theta_2). \tag{57}$$

We now make a change of variable in the above integral $\theta' = \theta_1 - \theta_2$, in terms of the new variable:

$$\vec{K}_{12n} = -\int_0^{2\pi} d\theta_2 \int_{-\theta_2}^{2\pi-\theta_2} d\theta' \cos(\theta') \left(\sqrt{3 - 2\cos(\theta')}\right)^{n-3} \vec{R}(\theta' + \theta_2, \theta_2). \tag{58}$$

In which:

$$\begin{aligned} \vec{R}(\theta' + \theta_2, \theta_2) \quad &= \quad ((\cos\theta' - 1)\cos\theta_2 - \sin\theta' \sin\theta_2, \\ &\quad\quad (\cos\theta' - 1)\sin\theta_2 + \sin\theta' \cos\theta_2, 1) \end{aligned} \tag{59}$$

Since the integrand is periodic with respect to $\theta'$ with a period of $2\pi$ we may write equation (58) as:

$$\vec{K}_{12n} = -\int_0^{2\pi} d\theta_2 \int_0^{2\pi} d\theta' \cos(\theta') \left(\sqrt{3 - 2\cos(\theta')}\right)^{n-3} \vec{R}(\theta' + \theta_2, \theta_2). \tag{60}$$

| n | $K_{z12n}$ |
|---|---|
| 0 | -7.18 |
| 1 | -6.74 |
| 2 | -4.94 |
| 3 | 0 |
| 4 | 11.97 |
| 5 | 39.48 |
| 6 | 101.03 |

Table 1: Table of the geometric factor $K$ values.

Changing the order of integrals and noticing that:

$$\int_0^{2\pi} \cos\theta_2 d\theta_2 = \int_0^{2\pi} \sin\theta_2 d\theta_2 = 0 \tag{61}$$

We arrive at the results:

$$\vec{K}_{12n} = \left(0, 0, -2\pi \int_0^{2\pi} d\theta' \cos(\theta') \left(\sqrt{3 - 2\cos(\theta')}\right)^{n-3}\right). \tag{62}$$

The vanishing of the x and y components of the force is to be expected due to the symmetry of the problem. The z components can be calculated analytically, but for even values of $n$ one needs to use elliptic functions, for example:

$$\begin{aligned} K_{z122} &= 4\pi(Ee(-4) - 3Ke(-4)) \simeq -4.94, \\ Ee(m) &\equiv \int_0^{\frac{\pi}{2}} d\theta \sqrt{1 - m\sin^2\theta} \\ Ke(m) &\equiv \int_0^{\frac{\pi}{2}} d\theta \frac{1}{\sqrt{1 - m\sin^2\theta}} \end{aligned} \tag{63}$$

The $K$ values up to $n = 6$ are given in table 1. We conclude this subsection by noticing that for multiple loops we have:

$$K_{z12n} \sim N_1 N_2. \tag{64}$$

$N_1$ and $N_2$ are the number of turns in coil 1 and coil 2 respectively.

# 4 Conclusion

We have shown in this paper that in general Newton's third law is not compatible with the principles of special relativity and the total force on a two current loop system is not zero.

As a final remark we will address the problem of achieving constant force which may be of interest for locomotive applications. A constant force may be achieved by having a direct current in one loop $I_1(t) = \bar{I}_1$ and a current

| Parameter | Value |
|---|---|
| $N_1$ | 1000 |
| $N_2$ | 1000 |
| $\bar{I}_1$ | 100 Ampere |
| $\bar{I}_2$ | 100 Ampere |
| $h$ | 0.1 meter |
| $\frac{h}{c}$ | 0.3 nano seconds |
| $\tau_c$ | 10 nano seconds |

Table 2: The choice of parameters for the force calculation.

of uniform second derivative on the other $I_2(t) = \frac{1}{2}\bar{I}_2\frac{t^2}{\tau_c^2}$. In this case the accelerating force will be according to equation (51):

$$F_{Tz} \cong -\frac{\mu_0}{8\pi}(\frac{h}{c})^2\frac{1}{\tau_c^2}K_{z122}\bar{I}_1\bar{I}_2 \tag{65}$$

Assuming the case of circular current loops of the previous subsection we have according to equation (63):

$$F_{Tz} \cong 4.94\frac{\mu_0}{8\pi}N_1N_2(\frac{h}{c})^2\frac{1}{\tau_c^2}\bar{I}_1\bar{I}_2 \tag{66}$$

For the choice of values given in table 2 we obtained $F_{Tz} \cong 2.74$ Newton. Obviously the switching time may represent some difficulty which one may overcome with advanced enough switching technology perhaps using low resistivity superconducting materials. Another possibility for constructing a relativistic motor is using numerous modular solid-state devices each with fast switching and small current such that an appreciable amount of cumulative forcing will result.

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